of Fermat as opposed to Descartes)—won out, as a result of the work of Euler, Lagrange, and Monge. Hence the term "analytic geometry" implied proof as well as the use of the algebraic method. Consequently we now speak of analytic geometry as opposed to synthetic geometry, and we no longer mean that one is a method of invention and the other of proof. Both are deductive.

In the meantime the calculus and extensions such as infinite series entered mathematics. Both Newton and Leibniz regarded the calculus as an extension of algebra; it was the algebra of the infinite, or the algebra that dealt with an infinite number of terms, as in the case of infinite series. As late as 1797, Lagrange, in *Théorie des fonctions analytiques*, said that the calculus and its developments were only a generalization of elementary algebra. Since algebra and analysis had been synonyms, the calculus was referred to as analysis. In a famous calculus text of 1748 Euler used the term "infinitesimal analysis" to describe the calculus. This term was used until the late nineteenth century, when the word "analysis" was adopted to describe the calculus and those branches of mathematics built on it. Thus we are left with a confusing situation in which the term "analysis" embraces all the developments based on limits, but "analytic geometry" involves no limit processes.

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KLINE

16
The Mathematization of Science

> So that we may say the door is now opened, for the first time, to a new method fraught with numerous and wonderful results which in future years will command the attention of other minds.
> GALILEO GALILEI

1. *Introduction*

By 1600 the European scientists were unquestionably impressed with the importance of mathematics for the study of nature. The strongest evidence of this conviction was the willingness of Copernicus and Kepler to overturn the accepted laws of astronomy and mechanics and religious doctrines for the sake of a theory which in their time had only mathematical advantages. However, the astonishing successes of modern science and the enormous impetus to creative work that mathematics derived from that source probably would not have come about if science had continued in the footsteps of the past. But in the seventeenth century two men, Descartes and Galileo, revolutionized the very nature of scientific activity. They selected the concepts science should employ, redefined the goals of scientific activity, and altered the methodology of science. Their reformulation not only imparted unexpected and unprecedented power to science but bound it indissolubly to mathematics. In fact, their plan practically reduced theoretical science to mathematics. To understand the spirit that animated mathematics from the seventeenth through the nineteenth centuries, we must first examine the ideas of Descartes and Galileo.

2. *Descartes's Concept of Science*

Descartes proclaimed explicitly that the essence of science was mathematics. He says that he "neither admits nor hopes for any principles in Physics other than those which are in Geometry or in abstract mathematics, because thus all phenomena of nature are explained and some demonstrations of them can be given." The objective world is space solidified, or geometry incarnate. Its properties should therefore be deducible from the first principle of geometry.

Descartes elaborated on why the world must be accessible and reducible to mathematics. He insisted that the most fundamental and reliable properties of matter are shape, extension, and motion in space and time. Since shape is just extension, Descartes asserted, "Give me extension and motion and I shall construct the universe." Motion itself resulted from the action of forces on molecules. Descartes was convinced that these forces obeyed invariable mathematical laws; and, since extension and motion were mathematically expressible, all phenomena were mathematically describable.

Descartes's mechanistic philosophy extended even to the functioning of the human body. He believed that laws of mechanics would explain life in man and animals, and in his work in physiology he used heat, hydraulics, tubes, valves, and the mechanical actions of levers to explain the actions of the body. However, God and the soul were exempt from mechanism.

If Descartes regarded the external world as consisting only of matter in motion, how did he account for tastes, smells, colors, and the qualities of sounds? Here he adopted the old Greek doctrine of primary and secondary qualities which, as stated by Democritus, maintained that "sweet and bitter, cold and warm, as well as the colors, all these things exist but in opinion and not in reality; what really exist are unchangeable particles, atoms, and their motions in empty space." The primary qualities, matter and motion, exist in the physical world; the secondary qualities are only effects the primary qualities induce in the sense organs of human beings by the impact of external atoms on these organs.

Thus to Descartes there are two worlds; one, a huge, harmoniously designed mathematical machine existing in space and time, and the other, the world of thinking minds. The effect of elements in the first world on the second produces the nonmathematical or secondary qualities of matter. Descartes affirmed further that the laws of nature are invariable, since they are but part of a predetermined mathematical pattern, and that God could not alter invariable nature. Here Descartes denied the prevailing belief that God continually intervened in the functioning of the universe.

Though Descartes's philosophical and scientific doctrines subverted Aristotelianism and Scholasticism, he was a Scholastic in one fundamental respect: he drew from his own mind propositions about the nature of being and reality. He believed that there are a priori truths and that the intellect by its own power may arrive at a perfect knowledge of all things; he stated laws of motion, for example, on the basis of a priori reasoning. (Actually in his biological work he did experiment, and he drew vital conclusions from the experiments.) However, apart from his reliance upon a priori principles, he did promulgate a general and systematic philosophy that shattered the hold of Scholasticism and opened up fresh channels of thought. His attempt to sweep away all preconceptions and prejudices was a clear declaration of revolt from the past. By reducing natural phenomena to purely physical

happenings he did much to rid science of mysticism and occult forces. Descartes's writings were highly influential; his deductive and systematic philosophy pervaded the seventeenth century and impressed Newton, especially, with the importance of motion. Daintily bound expositions of his philosophy even adorned ladies' dressing tables.

3. *Galileo's Approach to Science*

Though Galileo Galilei's philosophy of science agreed in large part with Descartes's, it was Galileo who formulated the more radical, more effective, and more concrete procedures for modern science and who by his own work demonstrated their effectiveness.

Galileo (1564–1642), born in Pisa to a cloth merchant, entered the University of Pisa to study medicine. The courses there were still at about the level of the medieval curriculum; Galileo learned his mathematics privately from a practical engineer, and at the age of seventeen switched from medicine to mathematics. After about eight years of study he applied for a teaching position at the University of Bologna but was refused as not sufficiently distinguished. He did secure a professorship of mathematics at Pisa. While there, he began to attack Aristotelian science; and he did not hesitate to express his views even though his criticisms alienated his colleagues. He had also begun to write important mathematical papers that aroused jealousy in the less competent. Galileo was made to feel uncomfortable and left in 1592 to accept the position of professor of mathematics at the University of Padua. There he wrote a short book, *Le mecaniche* (1604). After eighteen years at Padua he was invited to Florence by the Grand Duke Cosimo II de' Medici, who appointed him Chief Mathematician of his court, gave him a home and handsome salary, and protected him from the Jesuits, who dominated the papacy and had already threatened Galileo because he championed the Copernican theory. To express his gratitude, Galileo named the satellites of Jupiter, which he discovered in the first year of his service under Cosimo, the Medicean stars. In Florence Galileo had the leisure to pursue his studies and to write.

His advocacy of the Copernican theory irked the Roman Inquisition, and in 1616 he was called to Rome. His teachings on the heliocentric theory were condemned by the Inquisition; he had to promise not to publish any more on this subject. In 1630 Pope Urban VIII did give him permission to publish if he would make his book mathematical and not doctrinal. Thereupon, in 1632, he published his classic *Dialogo dei massimi sistemi* (Dialogue on the Great World Systems). The Roman Inquisition summoned him again in 1633 and under the threat of torture impelled him to recant his advocacy of the heliocentric theory. He was again forbidden to publish and required to live practically under house arrest. But he undertook to write up his years of thought and work on the phenomena of motion and on the strength of materials. The manuscript, entitled *Discorsi e dimostrazioni matematiche intorno à due nuove scienze* (Discourses and Mathematical Demonstrations Concerning Two New Sciences, also referred to as Dialogues Concerning Two New Sciences), was secretly transported to Holland and published there in 1638. This is the classic in which Galileo presented his new scientific method. He defended his actions with the words that he had never "declined in piety and reverence for the Church and my own conscience."

Galileo was an extraordinary man in many fields. He was a keen astronomical observer. He is often called the father of modern invention; though he did not invent the telescope or "perplexive glasses," as Ben Jonson called them, he was immediately able to construct one when he heard of the idea. He was an independent inventor of the microscope, and he designed the first pendulum clock. He also designed and made a compass with scales that automatically yielded the results of numerical computations so the user could read the scales and avoid having to do the calculations. This device was so much in demand that he produced many for sale.

Galileo was the first important modern student of sound. He suggested a wave theory of sound and began work on pitch, harmonics, and the vibrations of strings. This work was continued by Mersenne and Newton and became a major inspiration for mathematical work in the eighteenth century.

Galileo's major writings, though concerned with scientific subjects, are still regarded as literary masterpieces. His *Sidereus Nuncius* (Sidereal Messenger) of 1610, in which he announced his astronomical observations and declared himself in support of Copernican theory, was an immediate success, and he was elected to the prestigious Academy of the Lynx-like in Rome. His two greatest classics, the *Dialogue on the Great World Systems* and *Dialogues Concerning Two New Sciences*, are clear, direct, witty, yet profound. In both, Galileo has one character present the current views, against which another argues cleverly and tenaciously to show the fallacies and weaknesses of these views and the strengths of the new ones.

In his philosophy of science Galileo broke sharply from the speculative and mystical in favor of a mechanical and mathematical view of nature. He also believed that scientific problems should not become enmeshed in and beclouded by theological arguments. Indeed, one of his achievements in science, though somewhat apart from the method we are about to examine, is that he recognized clearly the domain of science and severed it sharply from religious doctrines.

Galileo, like Descartes, was certain that nature is mathematically designed. His statement of 1610 is famous:

> Philosophy [nature] is written in that great book which ever lies before our eyes—I mean the universe—but we cannot understand it if we do not first learn the language and grasp the symbols in which it is written. The book is written in the mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth.[1]

Nature is simple and orderly and its behavior is regular and necessary. It acts in accordance with perfect and immutable mathematical laws. Divine reason is the source of the rational in nature; God put into the world that rigorous mathematical necessity that men reach only laboriously. Mathematical knowledge is therefore not only absolute truth, but as sacrosanct as any line of Scripture. In fact it is superior, for there is much disagreement about the Scriptures, but there can be none about mathematical truths.

Another doctrine, the atomism of the Greek Democritus, is clearer in Galileo than in Descartes. Atomism presupposed empty space (which Descartes did not accept) and individual, indestructible atoms. Change consisted in the combination and separation of atoms. All qualitative varieties in bodies were due to quantitative variety in number, size, shape, and spatial arrangement of the atoms. The atom's chief properties were impenetrability and indestructibility; these properties served to explain chemical and physical phenomena. Galileo's espousal of atomism placed it in the forefront of scientific doctrines.

Atomism led Galileo to the doctrine of primary and secondary qualities. He says, "If ears, tongues, and noses were removed, I am of the opinion that shape, quantity [size] and motion would remain, but there would be an end of smells, tastes, and sounds, which abstracted from the living creature, I take to be mere words." Thus in one swoop Galileo, like Descartes, stripped away a thousand phenomena and qualities to concentrate on matter and motion, properties that are mathematically describable. It is perhaps not too surprising that in the century in which problems of motion were the most prominent and serious, scientists should find motion to be a fundamental physical phenomenon.

The concentration on matter and motion was only the first step in Galileo's new approach to nature. His next thought, also voiced by Descartes, was that any branch of science should be patterned on the model of mathematics. This implies two essential steps. Mathematics starts with axioms—clear, self-evident truths—and from these proceeds by deductive reasoning to establish new truths. Any branch of science, then, should start with axioms or principles and then proceed deductively. Moreover, one should extract from the axioms as many consequences as possible. This thought, of course, goes back to Aristotle, who also aimed at deductive structure in science with the mathematical model in mind.

However, Galileo departed radically from the Greeks, the medieval

1. *Opere*, 4, 171.

scientists, and even Descartes in his method of obtaining first principles. The pre-Galileans and Descartes had believed that the mind supplied the basic principles; it had but to think about any class of phenomena and it would immediately recognize fundamental truths. This power of the mind was clearly evidenced in mathematics. Axioms such as "equals added to equals give equals" and "two points determine a line" suggested themselves immediately in thinking about number or geometrical figures, and were indubitable truths. So too had the Greeks found some physical principles equally appealing. That all objects in the universe should have a natural place was no more than fitting. The state of rest seemed clearly more natural than the state of motion. It seemed indubitable, too, that force must be applied to put and keep bodies in motion. To believe that the mind supplies fundamental principles did not deny that observations might play a role in obtaining these principles. But the observations merely evoked the correct principles, just as the sight of a familiar face might call to mind facts about that person.

The Greek and medieval scientists were so convinced that there were a priori fundamental principles that when occasional observations did not fit they invented special explanations to preserve the principles but still account for the anomalies. These men, as Galileo put it, first decided how the world should function and then fitted what they saw into their preconceived principles.

Galileo decided that in physics, as opposed to mathematics, first principles must come from experience and experimentation. The way to obtain correct and basic principles is to pay attention to what nature says rather than what the mind prefers. Nature, he argued, did not first make men's brains and then arrange the world to be acceptable to human intellects. To the medieval thinkers who kept repeating Aristotle and debating what he meant, Galileo addressed the criticism that knowledge comes from observation and not from books, and that it was useless to debate about Aristotle. He says, "When we have the decrees of nature, authority goes for nothing...." Of course some Renaissance thinkers and Galileo's contemporary Francis Bacon had also arrived at the conclusion that experimentation was necessary; in this particular aspect of his new method, Galileo was not ahead of all others. Yet the modernist Descartes did not grant the wisdom of Galileo's reliance upon experimentation. The facts of the senses, Descartes said, can only lead to delusion, but reason penetrates such delusions. From the innate general principles supplied by the mind, we can deduce particular phenomena of nature and understand them. Galileo did appreciate that one may glean an incorrect principle from experimentation and that as a consequence the deductions from it could be incorrect. Hence he proposed the use of experiments to check the conclusions of his reasonings as well as to acquire basic principles.

Galileo was actually a transitional figure as far as experimentation is concerned. He, and Isaac Newton fifty years later, believed that a few key or critical experiments would yield correct fundamental principles. Moreover, many of Galileo's so-called experiments were really thought-experiments; that is, he relied upon common experience to imagine what would happen if an experiment were performed. He then drew a conclusion as confidently as if he had actually performed the experiment. When in the *Dialogue on the Great World Systems* he describes the motion of a ball dropped from the mast of a moving ship, he is asked by Simplicio, one of the characters, whether he had made an experiment. Galileo replies, "No, and I do not need it, as without any experience I can confirm that it is so, because it cannot be otherwise." He says in fact that he experimented rarely, and then primarily to refute those who did not follow the mathematics. Though Newton performed some famous and ingenious experiments, he too says that he used experiments to make his *results* physically intelligible and to convince the common people.

The truth of the matter is that Galileo had some preconceptions about nature, which made him confident that a few experiments would suffice. He believed, for example, that nature was simple. Hence when he considered freely falling bodies, which fall with increasing velocity, he supposed that the increase in velocity is the same for each second of fall. This was the simplest "truth." He believed also that nature is mathematically designed, and hence any mathematical law that seemed to fit even on the basis of rather limited experimentation appeared to him to be correct.

For Galileo, as well as for Huygens and Newton, the deductive, mathematical part of the scientific enterprise played a greater part than the experimental. Galileo was no less proud of the abundance of theorems that flow from a single principle than of the discovery of the principle itself. The men who fashioned modern science—Descartes, Galileo, Huygens, and Newton (we can also include Copernicus and Kepler)—approached the study of nature as mathematicians, in their general method and in their concrete investigations. They were primarily speculative thinkers who expected to apprehend broad, deep (but also simple), clear, and immutable mathematical principles either through intuition or through crucial observations and experiments, and then to deduce new laws from these fundamental truths, entirely in the manner in which mathematics proper had constructed its geometry. The bulk of the activity was to be the deductive portion; whole systems of thought were to be so derived.

What the great thinkers of the seventeenth century envisaged as the proper procedure for science did indeed prove to be the profitable course. The rational search for laws of nature produced, by Newton's time, extremely valuable results on the basis of the slimmest observational and experimental knowledge. The great scientific advances of the sixteenth and seventeenth centuries were in astronomy, where observation offered little that was new, and in mechanics, where the experimental results were hardly startling and certainly not decisive, whereas the mathematical theory attained comprehensiveness and perfection. And for the next two centuries scientists produced deep and sweeping laws of nature on the basis of very few, almost trivial, observations and experiments.

The expectation of Galileo, Huygens, and Newton that just a few experiments would suffice can be readily understood. Because these men were convinced that nature is mathematically designed, they saw no reason why they could not proceed in scientific matters much as mathematicians had proceeded in their domain. As John Herman Randall says in *Making of the Modern Mind*, "Science was born of a faith in the mathematical interpretation of nature. . . ."

Galileo did, however, obtain a few principles from experience; and in this work also his approach was a radical departure from that of his predecessors. He decided that one must penetrate to what is fundamental in phenomena and start there. In *Two New Sciences* he says that it is not possible to treat the infinite variety of weights, shapes, and velocities. He had observed that the speeds with which dissimilar objects fall differ less in air than in water. Hence the thinner the medium, the less difference in speed of fall among bodies. "Having observed this I came to the conclusion that in a medium totally devoid of resistance all bodies would fall with the same speed." What Galileo was doing here was to strip away the incidental or minor effects in an effort to get at the major one.

Of course, actual bodies do fall in resisting media. What could Galileo say about such motions? His answer was ". . . hence, in order to handle this matter in a scientific way, it is necessary to cut loose from these difficulties [air resistance, friction, etc.] and having discovered and demonstrated the theorems in the case of no resistance, to use them and apply them with such limitations as experience will teach."

Having stripped away air resistance and friction, Galileo sought basic laws for motion in a vacuum. Thus he not only contradicted Aristotle and even Descartes by thinking of bodies moving in empty space, but did just what the mathematician does in studying real figures. The mathematician strips away molecular structure, color, and thickness of lines to get at some basic properties and concentrates on these. So did Galileo penetrate to basic physical factors. The mathematical method of abstraction is indeed a step away from reality but, paradoxically, it leads back to reality with greater power than if all the factors actually present are taken into account at once.

Thus far Galileo had formulated a number of methodological principles, many of which were suggested by the approach mathematics had employed in geometry. His next principle was to use mathematics itself, but in a special way. Unlike the Aristotelians and the late medieval scientists, who had fastened upon qualities they regarded as fundamental and studied the acquisition and loss of qualities or debated the meaning of the qualities, Galileo proposed to seek *quantitative* axioms. This change is most important; we shall see the full significance of it later, but an elementary example may be useful now. The Aristotelians said that a ball falls because it has weight, and that it falls to the earth because every object seeks its natural place and the natural place of heavy bodies is the center of the earth. These principles are qualitative. Even Kepler's first law of motion, that the path of each planet is an ellipse, is a qualitative statement. By contrast, let us consider the statement that the speed (in feet per second) with which a ball falls is 32 times the number of seconds it has been falling, or in symbols, $v = 32t$. This is a quantitative statement about how a ball falls. Galileo intended to seek such quantitative statements as his axioms, and he expected to deduce new ones by mathematical means. These deductions would also give quantitative knowledge. Moreover, as we have seen, mathematics was to be his essential medium.

The decision to seek quantitative knowledge expressed in formulas carried with it another radical decision, though first contact with it hardly reveals its full significance. The Aristotelians believed that one of the tasks of science was to explain why things happened; explanation meant unearthing the causes of a phenomenon. The statement that a body falls because it has weight gives the effective cause of the fall and the statement that it seeks its natural place gives the final cause. But the quantitative statement $v = 32t$, for whatever it may be worth, gives no explanation of why a ball falls; it tells only how the speed changes with the time. In other words, formulas do not explain; they describe. The knowledge of nature Galileo sought was descriptive. He says in *Two New Sciences*, "The cause of the acceleration of the motion of falling bodies is not a necessary part of the investigation." More generally, he points out that he will investigate and demonstrate some of the properties of motion without regard to what the causes might be. Positive scientific inquiries were to be separated from questions of ultimate causation, and speculation as to physical causes was to be abandoned.

First reactions to this principle of Galileo are likely to be negative. Description of phenomena in terms of formulas hardly seems to be more than a first step. It would seem that the true function of science had really been grasped by the Aristotelians, namely, to explain why phenomena happened. Even Descartes protested Galileo's decision to seek descriptive formulas. He said, "Everything that Galileo says about bodies falling in empty space is built without foundation: he ought first to have determined the nature of weight." Further, said Descartes, Galileo should reflect on ultimate reasons. But we shall see clearly after a few chapters that Galileo's decision to aim for description was the deepest and most fruitful idea that anyone has had about scientific methodology.

Whereas the Aristotelians had talked in terms of qualities such as fluidity, rigidity, essences, natural places, natural and violent motion, and potentiality, Galileo chose an entirely new set of concepts, which, moreover, were measurable, so that their measures could be related by formulas. Some of them are: distance, time, speed, acceleration, force, mass, and weight. These concepts are too familiar to surprise us. But in Galileo's time they were radical choices, at least as fundamental concepts; and these are the ones that proved most instrumental in the task of understanding and mastering nature.

We have described the essential features of Galileo's program. Some of the ideas in it had been espoused by others; some were entirely original with him. But what establishes Galileo's greatness is that he saw so clearly what was wrong or deficient in the current scientific efforts, shed completely the older ways, and formulated the new procedures so clearly. Moreover, in applying them to problems of motion he not only exemplified the method but succeeded in obtaining brilliant results—in other words, he showed that it worked. The unity of his work, the clarity of his thoughts and expressions, and the force of his argumentation influenced almost all of his contemporaries and successors. More than any other man, Galileo is the founder of the methodology of modern science. He was fully conscious of what he had accomplished (see the chapter legend); so were others. The philosopher Hobbes said of Galileo, "He has been the first to open to us the door to the whole realm of physics."

We cannot pursue the history of the methodology of science. However, since mathematics became so important in this methodology and profited so much from its adoption, we should note how completely Galileo's program was accepted by giants such as Newton. He asserts that experiments are needed to furnish basic laws. Newton is also clear that the function of science, after having obtained some basic principles, is to deduce new facts from these principles. In the preface to his *Principia*, he says:

> Since the ancients (as we are told by Pappus) esteemed the science of mechanics of greatest importance in the investigation of natural things, and the moderns, rejecting substantial forms and occult qualities, have endeavored to subject the phenomena of nature to the laws of mathematics, I have in this treatise cultivated mathematics as far as it relates to philosophy [science] . . . and therefore I offer this work as the mathematical principles of philosophy, for the whole burden in philosophy seems to consist in this—from the phenomena of motions to investigate the forces of nature, and then from these forces to demonstrate the other phenomena. . . .

Of course, mathematical principles, to Newton as to Galileo, were quantitative principles. He says in the *Principia* that his purpose is to discover and set forth the exact manner in which "all things had been ordered in measure, number and weight." Newton had good reason to emphasize quantitative mathematical laws, as opposed to physical explanation, because the central physical concept in his celestial mechanics was the force of gravitation, whose action could not be explained at all in physical terms. In lieu of explanation Newton had a quantitative formulation of how gravity acted that was significant and usable. And this is why he says, at the beginning of the *Principia*, "For I here design only to give a mathematical notion of these forces, without considering their physical causes and seats." Toward the end of the book he repeats this thought:

> But our purpose is only to trace out the quantity and properties of this force from the phenomena, and to apply what we discover in some simple cases as principles, by which, in a mathematical way, we may estimate the effects thereof in more involved cases . . . We said, in *a mathematical way* [italics Newton's], to avoid all questions about the nature or quality of this force, which we would not be understood to determine by any hypothesis. . ."

The abandonment of physical mechanism in favor of mathematical description shocked even great scientists. Huygens regarded the idea of gravitation as "absurd," because its action through empty space precluded any mechanism. He expressed surprise that Newton should have taken the trouble to make such a number of laborious calculations with no foundation but the mathematical principle of gravitation. Leibniz attacked gravitation as an incorporeal and inexplicable power; John Bernoulli (James's brother) denounced it as "revolting to minds accustomed to receiving no principle in physics save those which are incontestable and evident." But this reliance on mathematical description even where physical understanding was completely lacking made possible Newton's amazing contributions, to say nothing of subsequent developments.

Because science became heavily dependent upon—almost subordinate to—mathematics, it was the scientists who extended the domain and techniques of mathematics; and the multiplicity of problems provided by science gave mathematicians numerous and weighty directions for creative work.

4. *The Function Concept*

The first mathematical gain from scientific investigations conducted in accordance with Galileo's program came from the study of motion. This problem engrossed the scientists and mathematicians of the seventeenth century. It is easy to see why. Though Kepler's astronomy was accepted early in the seventeenth century, especially after Galileo's observations supplied additional evidence for a heliocentric theory, Kepler's law of elliptical motion is only approximately correct, though it would be exact if there were just the sun and one planet in the heavens. The ideas that the other planets disturb the elliptical motion of any one planet and that the sun disturbs the elliptical motion of the moon around the earth were already being considered; in fact, the notion of a gravitational force acting between any two bodies was suggested by Kepler, among others. Hence the problem of improving the calculation of the planets' positions was open. Moreover, Kepler had obtained his laws essentially by fitting curves to astronomical data, with no explanation in terms of fundamental laws of motion of why the planets moved in elliptical paths. The basic problem of deriving Kepler's laws from principles of motion posed a clear challenge.

The improvement of astronomical theory also had a practical objective. In their search for raw materials and trade, the Europeans had undertaken large-scale navigation that involved sailing long distances out of sight of land. Mariners therefore needed accurate methods of determining latitude and longitude. The determination of latitude can be made by direct observation of the sun or the stars, but determination of longitude is far more difficult. In the sixteenth century the methods of doing it were so inaccurate that navigators were often in error as much as 500 miles. After about 1514, the direction of the moon relative to the stars was used to determine longitude. These directions, as seen from some standard place at various times, were tabulated. A navigator would determine the direction of the moon, which was not affected much by his being in a different location, and determine his local time by using, for example, the directions of the stars. Directly from the tables or by interpolation he could find the time at the standard location when the moon had the measured direction and so compute the difference in time between his position and the standard one. Each hour of difference means a 15-degree difference in longitude. This method, however, was not accurate. Because the ships of those times were constantly heaving, it was difficult to obtain the moon's direction accurately; but, because the moon does not move much relative to the stars in a few hours, the direction of the moon had to be rather precisely determined. A mistake of one minute of angle means an error of half a degree of longitude; but even a measure accurate to within one minute was far beyond the capabilities of those times. Though other methods of determining longitude were suggested and tried, better knowledge of the moon's path to extend and improve the tables seemed indispensable and many scientists, including Newton, worked on the problem. Even in Newton's time the knowledge of the moon's position was so inaccurate that use of the tables led to errors of as much as 100 miles in determining position at sea.

The governments of Europe were very much concerned, because

shipping losses were considerable. In 1675 King Charles II of England set up the Royal Observatory at Greenwich to obtain better observations on the moon's motion and to serve as a fixed station for longitude. In 1712 the British government established a Commission for the Discovery of Longitude and offered rewards of up to £20,000 for ideas on how to measure longitude.

The problem of explaining terrestrial motions also faced seventeenth-century scientists. Under the heliocentric theory the earth was both rotating and revolving around the sun. Why then should objects stay with the earth? Why should dropped objects fall to earth if it was no longer the center of the universe? Moreover, all motions, projectile motion for example, seemed to take place as though the earth were at rest. These questions engaged the attention of many men, including Cardan, Tartaglia, Galileo, and Newton. The paths of projectiles, their ranges, the heights they could reach, and the effect of muzzle velocity on height and range were basic questions and the princes then, like nations now, spent great sums on the solutions. New principles of motion were needed to account for these terrestrial phenomena; and it occurred to the scientists that, since the universe was believed to be constructed according to one master plan, the same principles that explained terrestrial motions would also account for heavenly motions.

From the study of the various problems of motion there emerged the specific problem of designing more accurate methods of measuring time. Mechanical clocks, which had been in use since 1348, were not very accurate. The Flemish cartographer Gemma Frisius (1508–55) had suggested the use of a clock to determine longitude. A ship could carry a clock set to the time of a place of known longitude; since the determination of local time by the sun's position, for example, was relatively simple, the navigator need merely note the difference in time and translate this at once into the difference in longitude. But no durable, accurate, seaworthy clocks were available even by 1600.

The motion of a pendulum seemed to provide the basic mechanism for measuring time. Galileo had observed that the time for one complete oscillation of a pendulum was constant and ostensibly independent of the amplitude of the swing. He prepared the design of a pendulum clock and had his son construct one; but it was Robert Hooke and Huygens who did the basic work on the pendulum. Though the pendulum clock was unsuitable for a ship (an accuracy of two or three seconds a day was needed for the purpose of longitude-reckoning, and pendulums were too much affected by ship's motion), it proved immensely valuable in scientific work, as well as for timekeeping in homes and business. A clock appropriate for navigation was finally designed by John Harrison (1693–1776) in 1761 and began to be used by the end of the eighteenth century. Because a proper clock was not available earlier, accurate determination of the motion of the moon was still the chief scientific problem in that century.

From the study of motion mathematics derived a fundamental concept that was central to practically all of the work for the next two hundred years —the concept of a function or a relation between variables. One finds this notion almost throughout Galileo's *Two New Sciences*, the book in which he founded modern mechanics. Galileo expresses his functional relationships in words and in the language of proportion. Thus in his work on the strength of materials, he has occasion to state, "The areas of two cylinders of equal volumes, neglecting the bases, bear to each other a ratio which is the square root of the ratio of their lengths." Again, "The volumes of right cylinders having equal curved surfaces are inversely proportional to their altitudes." In his work on motion he states, for example, "The spaces described by a body falling from rest with a uniformly accelerated motion are to each other as the squares of the time intervals employed in traversing these distances." "The times of descent along inclined planes of the same height, but of different slopes, are to each other as the lengths of these planes." The language shows clearly that he is dealing with variables and functions; it was but a short step to write these statements in symbolic form. Since the symbolism of algebra was being extended at this time, Galileo's statement on the spaces described by a falling body soon was written as $s = kt^2$ and his statement on times of descent as $t = kl$.

Most of the functions introduced during the seventeenth century were first studied as curves, before the function concept was fully recognized. This was true, for example, of the elementary transcendental functions such as $\log x$, $\sin x$, and a^x. Thus Evangelista Torricelli (1608–47), a pupil of Galileo, in a letter of 1644 described his research on the curve we would represent by $y = ae^{-cx}$ with $x \geq 0$ (the manuscript in which he wrote up this research was not edited until 1900). The curve was suggested to Torricelli by the current work on logarithms. Descartes encountered the same curve in 1639 but did not speak of its connection with logarithms. The sine curve entered mathematics as the companion curve to the cycloid in Roberval's work on the cycloid (Chap. 17, sec. 2) and appears graphed for two periods in Wallis's *Mechanica* (1670). Of course the tabular values of the trigonometric and logarithmic functions were, by this time, known with great precision.

It is also relevant that old and new curves were introduced by means of motions. In Greek times, a few curves, such as the quadratrix and the Archimedean spiral, were defined in terms of motion, but in that period such curves were outside the pale of legitimate mathematics. The attitude was quite different in the seventeenth century. Mersenne in 1615 defined the cycloid (which had been known earlier) as the locus of a point on a wheel that rolls along the ground. Galileo, who had shown that the path of a projectile shot up into the air at an angle to the ground is a parabola, regarded the curve as the locus of a moving point.

With Roberval, Barrow, and Newton the concept of a curve as the path of a moving point attains explicit recognition and acceptance. Newton says in *Quadrature of Curves* (written in 1676), "I consider mathematical quantities in this place not as consisting of very small parts, but as described by a continued motion. Lines [curves] are described, and thereby generated, not by the apposition of parts but by the continued motion of points.... These geneses really take place in the nature of things, and are daily seen in the motion of bodies."

Gradually the terms and symbolism for the various types of functions represented by these curves were introduced. There were many subtle difficulties that were hardly recognized. For example, the use of functions of the form a^x, with x taking on positive and negative integral and fractional values, became common in the seventeenth century. It was assumed (until the nineteenth century, when irrational numbers were first defined) that the function was also defined for irrational values of x, so that no one questioned an expression of the form $2^{\sqrt{2}}$. The implicit understanding was that such a value was intermediate between that obtained for any two rational exponents above and below $\sqrt{2}$.

Descartes's distinction between geometric and mechanical curves (Chap. 15, sec. 4) gave rise to the distinction between algebraic and transcendental functions. Fortunately his contemporaries ignored his banishment of what he called mechanical curves. Through quadratures, the summation of series, and other operations that entered with the calculus, many types of transcendental functions arose and were studied. The distinction between algebraic and transcendental functions was clearly made by James Gregory in 1667, when he sought to show that the area of a circular sector could not be an algebriac function of the radius and the chord. Leibniz showed that $\sin x$ cannot be an algebraic function of x and incidentally proved the result sought by Gregory.[2] The full understanding and use of the transcendental functions came gradually.

The most explicit definition of the function concept in the seventeenth century was given by James Gregory in his *Vera Circuli et Hyperbolae Quadratura* (1667). He defined a function as a quantity obtained from other quantities by a succession of algebraic operations or by any other operation imaginable. By the last phrase he meant, as he explains, that it is necessary to add to the five operations of algebra a sixth operation, which he defines as passage to the limit. (Gregory, as we shall see in Chapter 17, was concerned with quadrature problems.) Gregory's concept of function was lost sight of; but in any case, it would soon have proved too narrow, because the series representation of functions became widely used.

From the very beginning of his work on the calculus, that is from 1665

2. *Math. Schriften*, 5, 97–98.

on, Newton used the term "fluent" to represent any relationship between variables. In a manuscript of 1673 Leibniz used the word "function" to mean any quantity varying from point to point of a curve—for example, the length of the tangent, the normal, the subtangent, and the ordinate. The curve itself was said to be given by an equation. Leibniz also introduced the words "constant," "variable," and "parameter," the latter used in connection with a family of curves.[3] In working with functions John Bernoulli spoke from 1697 on of a quantity formed, in any manner whatever, of variables and of constants;[4] by "any manner" he meant to cover algebraic and transcendental expressions. He adopted Leibniz's phrase "function of x" for this quantity in 1698. In his *Historia* (1714), Leibniz used the word "function" to mean quantities that depend on a variable.

As to notation, John Bernoulli wrote X or ξ for a general function of x, though in 1718 he changed to ϕx. Leibniz approved of this, but proposed also x^1 and x^2 for functions of x, the superscript to be used when several functions were involved. The notation $f(x)$ was introduced by Euler in 1734.[5] The function concept immediately became central in the work on the calculus. We shall see later how the concept was extended.

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3. *Math. Schriften*, 5, 266–69.
4. *Mém de l'Acad des Sci.*, Paris, 1718, 100 ff. = *Opera*, 2, 235–69, p. 241 in particular.
5. *Comm. Acad. Sci. Petrop.*, 7, 1734/35, 184–200, pub. 1740 = *Opera*, (1), 22, 57–75.

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